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TECHNICAL MEMORANDUM 1382

STEADY PROPERLY-BANKED TURNS OF TURBOJET-PROPELLED AIRPLANES

By Angelo Miele

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SUMMARY

The problem of a jet-propelled airplane held in a steady turn is treated both in the very general case and also in the particular case when the polar curve can be approximated by a parabola.

Once the general solution has been obtained, some typical maneuvers are next studied such as, the turn of maximum bank, of maximum angular velocity, and of minimum radius of curvature.

After a brief comparison is made between the turning characteristics of conventional airplanes and jet airplanes, and after the effect of compressibility upon the turn is examined, the effects of the salient aerodynamic and structural parameters upon the behavior of the plane in curvilinear flight are summarized in the conclusions.

SYMBOLS

a	velocity of sound at the altitude Z, m/sec
C_p	lift coefficient
C_{p_e}	lift coefficient corresponding to the condition of maximum lift-to-drag ratio
$C_{p_{lim}}$	lift coefficient denoting the upper limit of the range within which it is permissible to approximate the experimental values by means of a parabolic polar
$C_{p_{max}}$	maximum lift coefficient

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C_r drag coefficient

C_{r_c} drag coefficient corresponding to the stall angle

C_{r_e} drag coefficient corresponding to the condition of maximum lift-to-drag ratio

C_{r_0} minimum drag coefficient

e airplane efficiency factor

E lift-to-drag ratio

E_{max} maximum lift-to-drag ratio

F_c centrifugal force, kg

g acceleration due to gravity, 9.807 m/sec^2

K ratio of specific heats, 1.4 for air

M Mach number at altitude Z

M_0 Mach number at sea level

M_* Mach number corresponding to the tropopause

n load factor, P/Q

p atmospheric pressure at the altitude Z, kg/m^2

P lift, kg

Q weight of the airplane, kg

r radius of turn

R aerodynamic drag, kg

S wing area, m^2

t ratio between the effective thrust and the minimum thrust required for straight and level steady flight, T/T_{min}

T thrust at altitude Z, kg

T_{min} minimum thrust required for steady horizontal flight, Q/E_{max}, kg

T_0	thrust at sea level, kg
T_*	thrust existing at the tropopause, kg
V	velocity, m/sec
V_e	velocity of straight horizontal flight at maximum lift-to-drag ratio, m/sec
V_{min}	stalling speed in straight horizontal flight, m/sec
x	ratio between the effective coefficient of lift and the coefficient of lift corresponding to the maximum lift-to-drag ratio, C_p/C_{p_e}
α	angle of incidence
α_e	angle of attack corresponding to the condition of maximum lift-to-drag ratio
θ	slope of the plane's path
λ	geometric aspect ratio of the wing
ρ	absolute density of the air at the altitude Z, kg sec^2 m^{-4}
σ	relative density of the air at the altitude Z
σ_*	relative density of the air at the tropopause
τ	absolute temperature of the air at an altitude Z, $^\circ$K
φ	lateral inclination (bank) angle, angle between the plane of symmetry of the airplane and the vertical plane containing the tangent to the plane's path
ψ	angle of yaw
ω	angular velocity, V/r, sec^{-1}

1. GENERAL CONSIDERATIONS

It is well known that the most general type of uniform flight of an airplane is a helical motion with respect to the vertical axis, executed at constant velocity and constant angular velocity.

It is also known that, for given operating conditions of the propulsion mechanism and for an ideal atmosphere of constant density, there exist an infinite number of possible helical motions each being defined by three independent parameters. Each one of these quantities is determined if three of the independent parameters which characterize the motion are assigned (naturally, they must be selected within the limits of physical possibilities for flight).

A particular case of helical flight is the properly banked turn at a constant altitude.

The assumption that such a maneuver is to be executed at a constant altitude is equivalent to fixing one of the three independent parameters of the helical motion, because the angle of inclination that the tangent to the flight path makes with respect to the horizontal plane must vanish ($\theta = 0$).

The other hypothesis, that the airplane should be properly banked, is equivalent to supposing that the vector representing the velocity of the center of gravity lies entirely within the plane of symmetry of the airplane. Consequently, the angle of yaw vanishes ($\psi = 0$).

To have fixed right at the start two of the three independent parameters which determine the helical motion of the airplane implies that, for a given regime of operation of the turbojet, the airplane can execute an infinite number of single parameter properly-banked horizontal turns, whose characteristics will be single-valued functions of the angle of bank or of the load factor.

2. EQUATIONS OF MOTION OF THE CENTER OF GRAVITY

The general vectorial condition for equilibrium of the forces acting during a steady, properly-banked turn may be written as:

$$\vec{T} + \vec{R} + \vec{P} + \vec{Q} + \vec{F}_c = 0 \tag{1}$$

Equation (1) is equivalent to three scalar equations, which are materially simplified in the case where we use, as coordinate axes, the principle axes of the flight path (the tangent, the principal normal, and the binormal to the flight path). In fact, upon projection of equation (1) upon the three above-mentioned axes, we obtain:

$$T - R = 0 \tag{2}$$

$$F_c - P \sin \varphi = 0 \tag{3}$$

$$Q - P \cos \varphi = 0 \tag{4}$$

These preceding equations are transformed into the following expressions when the well-known relationships for the lift, the drag, and the centrifugal force are utilized:

$$T - \frac{1}{2} C_r \rho S V^2 = 0 \tag{5}$$

$$\frac{Q}{g} \frac{V^2}{r} - \frac{1}{2} C_p \rho S V^2 \sin \varphi = 0 \tag{6}$$

$$Q - \frac{1}{2} C_p \rho S V^2 \cos \varphi = 0 \tag{7}$$

As is easily seen, the drag is balanced by the propulsive action of the turbojet, while the lift directly opposes the vector sum of the centrifugal force and the weight.

The aerodynamic coefficients which appear in equations (5), (6), and (7) are not independent from one another, but they are linked together through the drag polar. For the case of incompressible flow, this connection is then

$$C_r = C_r(C_p) \tag{8}$$

or

$$C_r = C_{r_0} + \frac{{C_p}^2}{\pi \lambda e} \tag{9}$$

according to whether one works with an experimental polar or with an assumed parabola.

At large angles of attack, it is indispensable to base the work on equation (8), it being taken for granted that the parabolic polar, by its very nature, will not be of proper shape to match the entire experimental curve.

The thrust of the turbojet, for a given r.p.m. and for a selected altitude, is a function of the velocity. On that account, it is true that

$$T = T(V) \tag{10}$$

It is necessary to note, however, that very often, especially in order to perform rapid design calculations, it is legitimate to neglect the dependence and to consider the propulsive action to be constant.

It is important to observe that equations (5), (6), (7), (10) and the equation of the polar constitute a system of 5 equations in six variables:

$$C_p, \quad C_r, \quad V, \quad T, \quad r, \quad \varphi \tag{11}$$

The solution is possible if one selects as given, any one of the parameters (11) whatsoever. This fact confirms what was said in the preceding paragraph about the existence of a single infinity of possible turns.

In addition to the quantities already mentioned, the load factor and the angular velocity are of interest; they are expressed, respectively, by:

$$n = \frac{P}{Q} \tag{12}$$

$$\omega = \frac{V}{r} \tag{13}$$

3. SOLUTION OF THE TURN-EQUATIONS FOR THE CASE OF A NONPARABOLIC POLAR

It is particularly convenient, as far as carrying out the calculations in a practical case is concerned, to select the velocity as the fundamental variable, especially if the propulsive action depends on this parameter.

If it is legitimate to make the assumption instead that

$$\frac{\partial T}{\partial V} \cong 0 \tag{14}$$

it is convenient to employ the angle of attack as independent variable.

The procedure which is best to adopt in the most general case is as follows:

A. For a given velocity, one derives the corresponding thrust from equation (10); that is, from the operation charts of the turbojet.

B. The quantities V and T now being known, equation (5) allows us to find the drag coefficient

$$C_r = \frac{2T}{\rho S V^2} \tag{15}$$

C. From the airplane's polar, one may easily obtain that

$$C_p = f(C_r) \tag{16}$$

$$E = \frac{P}{R} = \frac{C_p}{C_r} \tag{17}$$

D. By means of equations (2), (12), and (17), one deduces the expression for the load factor

$$n = \frac{TE}{Q} \tag{18}$$

E. The bank angle is determined through use of equations (4) and (12) as:

$$\varphi = \text{arc cos}\,\frac{1}{n} = \text{arc cos}\,\frac{Q}{TE} \tag{19}$$

or also, recalling a simple trigonometric relationship, it is obtained from

$$\varphi = \text{arc tan}\sqrt{n^2 - 1} = \text{arc tan}\sqrt{\left(\frac{TE}{Q}\right)^2 - 1} \tag{20}$$

F. The radius of curvature of the turn is found by use of equations (6) and (7) as:

$$r = \frac{V^2}{g \tan \varphi} \tag{21}$$

and this expression, when transformed on the basis of the relationships given as equations (15), (17) and (20) becomes:

$$r = \frac{2Q}{\rho g S \sqrt{C_p^2 - \left(\frac{Q}{T} C_r\right)^2}} \tag{22}$$

G. The angular velocity can be calculated easily from equations (13) and (21) through use of the expression:

$$\omega = \frac{g \tan \varphi}{V} \tag{23}$$

which, upon utilization of the equations (15), (17) and (20) produces:

$$\omega = g\sqrt{\frac{\rho S}{2Q}}\sqrt{\frac{T}{Q}\frac{C_p^2}{C_r} - \frac{Q}{T} C_r} \tag{24}$$

The preceding explicit formulae show that for each given velocity and for each given lift coefficient, there exists a corresponding turn. Such flight degenerates, as a particular case, to the condition of straight-line motion at a constant altitude if it so happens that

$$r = \infty \tag{25}$$

that is, if it happens that

$$E = \frac{C_p}{C_r} = \frac{Q}{T} \tag{26}$$

according to equations (20) and (21).

There exist two values of the angle of attack which satisfy equation (26). Let us denote these values by the symbols α_1 and α_2. The steady turn at a constant altitude and with a finite radius of curvature is possible only when the angle of attack lies between these values, or provided

$$\alpha_1 < \alpha < \alpha_2 \tag{27}$$

4. TYPICAL FEATURES OF A TURN IN THE CASE OF A NONPARABOLIC POLAR

Out of the totality of lift coefficients which satisfy equation (27), it is instructive to study those which bring about some particular kinds of turn, that is,

A. A turn with maximum bank.

B. A turn with maximum angular velocity.

C. A turn with minimum radius of curvature.

For the purpose of simplifying the derivation of the formulae, let us introduce two convenient parameters:

A. The straight and level flight velocity for maximum lift-to-drag ratio, which is

$$V_e = \sqrt{\frac{2Q}{C_{p_e} \rho S}} \tag{28}$$

B. The ratio between the effective thrust and the minimum thrust required for straight and level steady flight, which is

$$t = \frac{T}{T_{min}} = \frac{TE_{max}}{Q} \tag{29}$$

4.1. Maximum-Bank Turn

Let us assume that relationship (14) is acceptable. Then as can be seen from examination of equation (19) or of equation (20), the condition that

$$\varphi = \varphi_{max} \tag{30}$$

is satisfied when the attitude of flight is that for maximum lift-to-drag ratio, that is, for the case where

$$E = E_{max} \tag{31}$$

In such a case as this, the quantities V, n, φ, ω, r take on the values, respectively, of

$$V = V_e \sqrt{t} \tag{32}$$

$$n = t \tag{33}$$

$$\tan \varphi = \sqrt{t^2 - 1} \tag{34}$$

$$\omega = \frac{g}{V_e}\sqrt{t - \frac{1}{t}} \tag{35}$$

$$r = \frac{V_e^2}{g\sqrt{1 - \frac{1}{t^2}}} \tag{36}$$

These relationships follow directly from the equations (15), (17), (18), (20), (22), (24), (28), (29) and (31).

For large values of the load factor (at low altitude, sizeable values of the thrust permit such large values of the load factor), i.e., in the case where

$$\frac{1}{n^2} \ll 1 \tag{37}$$

equations (34), (35), and (36) produce the approximate formulae given below, respectively:

$$\varphi \cong \text{arc tan } t \tag{38}$$

$$\omega \cong \frac{g}{V_e}\sqrt{t} \tag{39}$$

$$r \cong \frac{V_e^2}{g} \tag{40}$$

It is worthy of note, particularly, that the radius of the turn under the condition of maximum bank angle is identical, under the assumption made in equation (37), with twice the value of the kinematic height corresponding to the velocity of a straight and level flight made in the attitude of maximum lift-to-drag ratio.

4.2. Maximum-Angular-Velocity Turn

Expression (24) shows that this maximum angular velocity turn occurs when the function

$$f(C_p) = \frac{T}{Q}\frac{{C_p}^2}{C_r} - \frac{Q}{T}C_r \tag{41}$$

becomes a maximum.

In the case where it is legitimate to take the thrust to be independent of the velocity, it is easy to see that the maximum of equation (41), taking into account the relationship (18), is obtained for that particular value of the lift coefficient which satisfies the relationship

$$\frac{dC_p}{dC_r} = \frac{C_p}{C_r}\frac{1 + n^2}{2n^2} \tag{42}$$

Because, for $n \geq 1$, it is true that

$$\frac{1 + n^2}{2n^2} \leq 1 \tag{43}$$

one may deduce that the angle of attack which gives maximum angular velocity is one such that $C_p \geq C_{pe}$.

In the case where the load factor is of such magnitude that the condition expressed by equation (37) can be considered acceptable, the relation given as equation (42) produces the approximation

$$\frac{dC_p}{dC_r} \cong \frac{C_p}{2C_r} \tag{44}$$

which corresponds to the condition that one obtains a maximum for the function[1]

$$f(C_p) = \frac{C_p}{\sqrt{C_r}} = E\sqrt{C_r} \tag{45}$$

By means of equations (18), (24), and (37), one can then derive the approximate expression for the maximum angular velocity as

$$\omega_{max} \cong \frac{g}{Q}\sqrt{\frac{T\rho S}{2}\left(E\sqrt{C_r}\right)_{max}} \tag{46}$$

4.3. Minimum-Radius-of-Curvature Turn

It is evident from equation (22) that the turn at minimum radius should be executed by use of an angle of attack which corresponds to the condition for which

$$f(C_p) = {C_p}^2 - \left(\frac{Q}{T}\,C_r\right)^2 \tag{47}$$

is a maximum.

The thrust being assumed independent of the velocity, and thus also independent of the attitude, it follows that the maximum of equation (47), upon taking into account the relationship adduced as equation (18), will be attained for that particular value of the lift coefficient which satisfies the relation

$$\frac{dC_p}{dC_r} = \frac{C_p}{C_r}\,\frac{1}{n^2} \tag{48}$$

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[1]In general, such a maximum exists on the real polar, and it is always situated, as is easily verified, in that region of the airplane's polar for which equation (9) is not applicable.

Since, for the case $n \geq 1$, we have

$$\frac{1}{n^2} \leq 1 \tag{49}$$

the proof is provided that the angle of attack for minimum radius of curvature is located in that region of the polar which corresponds to angles of incidence which are greater than the one for maximum lift-to-drag ratio.

As the value of T becomes larger, the angle of attack for minimum radius is displaced continuously towards the region of high incidences on the polar, thus continually approaching the stall angle.

So that, an approximate formula for calculation of the minimum radius of curvature, which is valid above all at low altitude and for sizeable values of the thrust, is:

$$r_{min} \cong \frac{2Q}{\rho g S \sqrt{C_{p_{max}}{}^2 - \left(\frac{Q}{T} C_{r_c}\right)^2}} \tag{50}$$

If, then, hypothesis (37) is satisfied, equation (50) produces, as a further simplification:

$$r_{min} \cong \frac{2Q}{\rho g S C_{p_{max}}} = \frac{V_{min}{}^2}{g} \tag{51}$$

In other words, the minimum radius coincides approximately with the value of twice the kinematic height corresponding to the velocity at the stall for the airplane in straight and level flight.

It is also worthy of note that the velocity for minimum radius of turn (V_A) is always less than the velocity for maximum angular velocity turn (V_B). (See fig. 1).

In fact, the condition that $r = r_{min}$ is obtained when the angle of inclination, α, of a straight line drawn through the origin of the coordinate system and any point on the curve representing the function $\omega = f(V)$ takes on its maximum value (when it is tangent to the curve).

Consequently, the ratio of lift to drag of the airplane flying so as to execute a minimum radius curve is less than the ratio of lift to drag which it exhibits during flight at ω_{max}; and thus, also, the same may be said in regard to the load factor, through use of equation (18).

From this, it follows that the assumption (37) is closer to what really occurs during a flight with maximum angular velocity, and for that reason, the approximation inherent in equation (46) is without doubt better than the approximation made in equation (51).[2]

Finally, one cannot help but make mention of the fact that scrutiny of equation (51) suggests the idea of improving the tightness of the turn by use of high lift devices.

Because of the approximations imposed in the derivation of such a formula one cannot be absolutely certain of such a statement. But it is necessary to examine ad hoc how the changes of the aerodynamic polar, attendant to the use of flaps, influence the maxima obtained from equations (41) and (47).

4.4. Comparison Between the Attitudes for Best Turn With Reciprocating Engine Propulsion and Those Conditions Giving Optimum Turns With Jet Propulsion

In the case of propulsion by reciprocating engines, provided it is assumed that the propeller efficiency is constant with variation in velocity, and that thus, the thrust decreases inversely with an increase in V, the optimum attitudes will be attained when the following conditions are satisfied.

A. Maximum-bank turn:

$$\frac{dC_p}{dC_r} = \frac{2}{3}\frac{C_p}{C_r} \tag{52}$$

B. Maximum-angular-velocity turn:

$$\frac{dC_p}{dC_r} = \frac{2}{3}\frac{C_p}{C_r}\frac{1 + n^2}{2n^2} \tag{53}$$

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[2]In all events, especially for low altitudes and sizeable values of the thrust, it is found that equation (51) can be utilized for the rapid estimation of the minimum radius turn, with a degree of accuracy which, on the average, will be within the limits of 10 to 15 percent of the correct result.

C. Minimum-radius-of-curvature turn:

$$\frac{dC_p}{dC_r} = \frac{2}{3}\frac{C_p}{C_r}\frac{1}{n^2} \tag{54}$$

Thus, it may be concluded that:

A. For propulsion by means of reciprocating engines, the attitude of the plane which produces the turn of maximum bank angle is identical with that attitude at which maximum endurance is obtained: $\left(E\sqrt{C_p}\right)_{max}$.

B. The attitude for attaining the turn with maximum angular velocity and the one holding for the condition of minimum radius of curvature (reciprocating engine installation) are at higher incidence than that for maximum endurance.

C. When the ratio of thrust/weight is the same, the attitudes which are optimum for the case of reciprocating engine installations are always at higher incidences than the attitudes for optimum turn performance for the jet propulsion installation.

4.5. Numerical Example

For the purpose of making more clear the nature of the considerations evolved above, the characteristics of a turning jet-propelled airplane have been calculated on the basis of the following data:

$Q = 5000$ kg	$\lambda = 5$
$S = 25$ sq. meters	$e = 0.8$
$C_{r_0} = 0.018$	$T_0 = 2100$ kg

In figure 2, the aerodynamic characteristics of the airplane are depicted.

In this graph, the following functions are illustrated: $E = f_1(C_p)$; $E = \sqrt{C_r} = f_2(C_p)$; $\frac{1}{E}\frac{dC_p}{dC_r} = f_3(C_p)$; besides $C_r = f(C_p)$.

In figure 3, the following quantities are shown diagramatically:

(1) The function $\omega = f(Z,V)$

(2) The curves which are the loci of the points in the (ω,V)-plane, corresponding to the conditions of ω_{max}, $C_{p_{max}}$, and $E\sqrt{C_r}_{max}$.

In figure 4, the following quantities are graphed:

(1) The function $r = f(Z,V)$.

(2) The curves which are the loci of the points in the (r,V)-plane corresponding to the condition of r_{min} and $C_{p_{max}}$.

Finally, in figure 5 are depicted, as functions of the altitude, the minimum radius of turn, the maximum angular velocity, and the maximum angle of bank. As is easily seen, the minimum radius of turn increases as the altitude increases, while ω_{max} and φ_{max} diminish as Z increases.

5. SOLUTION OF THE EQUATIONS FOR THE TURN IN THE CASE OF A PARABOLIC POLAR

If the polar of the airplane is expressible in parabolic form, the use of the graphico-numerical method can be avoided, because it is then possible to deal with a purely analytic procedure:

Let us denote the following quantities by

$$C_{p_e} = \sqrt{\pi\lambda e C_{r_0}} \tag{55}$$

$$C_{r_e} = 2C_{r_0} \tag{56}$$

$$E_{max} = \sqrt{\frac{\pi\lambda e}{4C_{r_0}}} \tag{57}$$

They are the aerodynamic coefficients which define the condition of maximum lift-to-drag ratio. Besides, let us introduce the quantity

$$x = \frac{C_p}{C_{p_e}} \tag{58}$$

which is the ratio between the lift coefficient experienced in any flight and the lift coefficient which corresponds to the condition of maximum lift-to-drag ratio. The polar of the airplane, as given in equation (9), takes on the new formulation

$$C_r = C_{r_0}(1 + x^2) \tag{59}$$

while the other parameters characterizing the turn, turn out to be

$$V = V_e\sqrt{\frac{2t}{1 + x^2}} \tag{60}$$

$$E = \frac{2x}{1 + x^2} E_{max} \tag{61}$$

$$n = \frac{2tx}{1 + x^2} \tag{62}$$

$$\tan \varphi = \sqrt{\left(\frac{2tx}{1 + x^2}\right)^2 - 1} \tag{63}$$

$$\omega = \frac{g}{V_e}\sqrt{\frac{2tx^2}{1 + x^2} - \frac{1 + x^2}{2t}} \tag{64}$$

$$r = \frac{V_e^2}{g\sqrt{x^2 - \left(\frac{1 + x^2}{2t}\right)^2}} \tag{65}$$

when the original expressions of equations (15), (17), (18), (20), (22), and (24) are transformed by aid of equations (28), (29), (55), (56), (57), (58) and (59).

6. CHARACTERISTIC FEATURES OF A TURN IN THE CASE OF A PARABOLIC POLAR

Upon assuming that equation (14) is true, it is now desired that the values of the lift coefficient be deduced which will produce the maximum values for φ, ω, and a minimum value for r, respectively.

Simple manipulations of equations (63), (64), and (65) show that the desired conditions are realized for

$$(x)_{\varphi_{max}} = 1 \tag{66}$$

$$(x)_{\omega_{max}} = \sqrt{2t - 1} \tag{67}$$

$$(x)_{r_{min}} = \sqrt{2t^2 - 1} \tag{68}$$

respectively.

As is easily seen, although the angle of attack for obtaining the turn with maximum bank is determined by factors which are solely aerodynamic, the angle of attack for flight at maximum angular velocity and with minimum radius of curvature each depend upon the thrust/weight ratio in addition. For that matter, this was already demonstrated in section 4.

Upon combining in order the expressions given as equations (63), (64) and (65) with the relationships (66), (67) and (68), one gets:

$$\varphi_{max} = \text{arc tan} \sqrt{t^2 - 1} \tag{69}$$

$$\omega_{max} = \frac{g}{V_e} \sqrt{2(t - 1)} \tag{70}$$

$$r_{min} = \frac{V_e^2}{g\sqrt{t^2 - 1}} \tag{71}$$

With regard to the use of equations (70) and (71), it is necessary to take heed that the said formulae are subject to restrictions. In fact, it is necessary to make sure by checking, whether the corresponding lift coefficients are below the finite limit of the lift coefficient up through which value the parabolic polar sticks closely to the experimental one.

Thus, it is necessary that

$$(C_p)_{\omega_{max}} = \sqrt{\pi\lambda e C_{r_0}(2t - 1)} \leq C_{p_{lim}} \tag{72}$$

and

$$(C_p)_{r_{min}} = \sqrt{\pi\lambda e C_{r_0}(2t^2 - 1)} \leq C_{p_{lim}} \tag{73}$$

be true.

If such inequalities are not satisfied, the formulae just written above lose their validity, and it is necessary to have recourse to the graphico-numerical procedures described in sections 3 and 4. From equations (72) and (73), it is easy to deduce that equations (70) and (71) become closer to actuality as t becomes smaller, and thus, for a given airplane and for a given r.p.m. of the turbine, they become closer to actuality as flight is made close to the absolute ceiling.

The theoretical absolute ceiling corresponds to the conditions:

$$\left.\begin{aligned} t &= 1 \\ \varphi_{max} &= 0 \\ \omega_{max} &= 0 \\ r_{min} &= \infty \end{aligned}\right\} \tag{74}$$

Again, in the case of a parabolic polar, the results are compiled into table I, wherein will be found described the following:

(1) The general formulae for a turn

(2) The formulae which apply to special cases of curvilinear flight

(3) The formulae which are connected with the case of straight and level flight, both with a given thrust, and with a minimum thrust.

On the other hand, the charts (6), (7), (8), (9), (10) and (11) are graphs for the three special cases of turn in which examination is made of the quantities:

$$x, \quad \frac{V}{V_e}, \quad n, \quad \varphi, \quad \frac{\omega V_e}{g}, \quad \frac{rg}{V_e^2}$$

as functions solely of "t", the ratio between the effective thrust and the minimum thrust necessary for steady straight and level flight.

7. THE EFFECT OF COMPRESSIBILITY ON THE TURN

As is indicated in table I, the velocities at which ω_{max} and r_{min} are attained in a turn increase as the altitude increases, while the corresponding angle of attack decreases when Z increases.

One deduces, therefore, that for a given kind of turn, whether we consider the Mach number of flight or if we mean the critical Mach number associated with the lift coefficient being employed, they both increase with increase in altitude, Z.

The presence of compressibility effects depends, substantially, on the laws of distribution of the above-mentioned Mach number as a function of altitude.

Upon assuming the following laws of variation for the thrust, for a given r.p.m. of the turbine, hold:

(a) For the troposphere: $$\frac{T}{T_0} = \sigma^{0.7} \qquad (75)$$

(b) For the stratosphere: $$\frac{T}{T_*} = \frac{\sigma}{\sigma_*} \qquad (76)$$

and upon taking into account the formulae of table I and the expression for the velocity of sound, it is easy to deduce the following relationships for the Mach number of flight:

A. The turn with maximum angle of bank

In the troposphere: $$\frac{M}{M_0} = \sigma^{-0.267} \qquad (77)$$

In the stratosphere: $$\frac{M}{M_*} = 1 \qquad (78)$$

B. The turn with maximum angular velocity

In the troposphere: $$\frac{M}{M_0} = \sigma^{-0.617} \qquad (79)$$

In the stratosphere: $$\frac{M}{M_*} = \sqrt{\frac{\sigma_*}{\sigma}} \qquad (80)$$

C. <u>The turn with minimum radius of curvature</u>

In the troposphere: $\frac{M}{M_*} = \sigma^{-0.967}$ (81)

In the stratosphere: $\frac{M}{M_*} = \frac{\sigma_*}{\sigma}$ (82)

In each case, if there are effects of compressibility operating, the procedure to use is the following:

(1) At an assigned altitude and for a given value of the velocity, the thrust is determined (where $T = T(V,Z)$) from the graphs of the performance charts of the jet engine, and the Mach number of flight is obtained from

$$M = \frac{V}{a} \cong 0.05 \frac{V}{\sqrt{T}} \tag{83}$$

(2) The drag coefficient is calculated from the relation

$$C_r = \frac{2T}{\rho SV^2} = \frac{2T}{K p S M^2} \tag{84}$$

(3) From the graph of the aerodynamic characteristics of the airplane, one determines the lift coefficient

$$C_p = f(C_r, M) \tag{85}$$

(4) Once the lift-to-drag ratio, E, has been calculated, the quantities n, φ, r, and ω are found by means of equations (18), (20), (21) and (13).

In general, one can predict that

A. The effects of compressibility are more likely to appear in the cases of turn under study, for airplanes with high wing loading, low value of minimum drag, and for small aspect ratio.

B. Upon taking into consideration that, for a given value of the lift coefficient, the compressibility effects manifest themselves by an increase in the drag coefficient, it follows that they decrease the maxima of the functions given by equations (17), (41) and (47).

Compressibility effects act, therefore, in a disadvantageous way upon the characteristics of the turn: they induce a decrease in the maximum bank angle and in the maximum angular velocity, and besides, they produce an increase in the minimum radius of curvature.

8. CONCLUSIONS

The study of a jet-propelled airplane held in a steady properly-banked turn, turns out to be more simple than the analogous study in the case of a conventionally-powered airplane. This is so because the assumption is made that the thrust is independent of the velocity, which permits a great simplification in the calculations, particularly in the case where the polar is to be described by means of a parabola.

The most significant deductions can be condensed into the following outline:

1. The characteristic quantities describing the properly-banked turn are single-valued functions of the angle of attack, or what is the same thing, double valued functions of the angle of bank.

2. Of the two turns which are generally possible at a given angle of bank, one belongs to the initial region of the polar $(\alpha \leq \alpha_e)$, while the other belongs to the region of high angles of attack $(\alpha \geq \alpha_e)$. The second is the region which produces the cases of a turn at smallest radii of curvature and at largest angular velocities.

3. There exist three special angles of attack which correspond, respectively, to the conditions of turns at φ_{max}, at ω_{max}, and at r_{min}. These angles of attack are characterized by the inequalities:

$$\alpha_e = (\alpha)_{\varphi_{max}} \leq (\alpha)_{\omega_{max}} \leq (\alpha)_{r_{min}} \tag{86}$$

4. The attitude for the turn at maximum angle of bank depends solely upon the aerodynamic characteristics of the airplane. On the contrary, the angles of incidence which characterize the turns at maximum angular velocity and with minimum radius of curvature depend, in addition, upon the ratio of thrust to weight.

5. The turn at maximum angle of bank is the one which produces the most structural stress upon the framework. The turns made at maximum angular velocity and at minimum radius of curvature produce load factors which are less, in that order.

6. For very large values of T/Q, the attitude giving the turn with maximum angular velocity, ω_{max}, occurs in close neighborhood to the condition corresponding to the maximum of the function,

$$f(C_p) = E\sqrt{C_r}$$

while the attitude giving the turn with minimum radius, r_{min}, becomes established when flying approximately with an angle of incidence which is equal to the stall angle.

7. The bank angle and the maximum angular velocity increase with an increase of the thrust and of the aspect ratio, and with a decrease in the minimum value of the drag coefficient and of the wing loading.

8. The minimum radius of turn diminishes as T and λ increase, and with a diminution in C_{r_0}, and $\frac{Q}{S}$.

9. The maximum bank angle and the maximum angular velocity decrease, for a given operating condition of the propulsion mechanism, with an increase in the flight altitude, finally vanishing in correspondence with flight at the absolute ceiling. The minimum radius of curvature, on the contrary, increases as Z increases, and tends gradually to an infinitely large value as the airplane approaches the absolute ceiling.

10. At a given ratio between the thrust and weight, the optimum angles of attack for turns with jet propulsion installations are always less than the ones made with reciprocating engine installations. These latter attitudes of flight, by the same token, belong to that region of the polar characterized by angles of incidence which are equal to or larger than the one for best endurance.

11. In connection with the effect of flaps on the characteristics of the turn, it does not appear to be a good idea to draw any conclusions of a general nature, but, taking each case on its own merits in view of the changes effected in the polar by the use of these devices, it is worth examining the advantages to be obtained from their use.

12. The compressibility effect, when present, acts in such a way as to be disadvantageous, insofar as the characteristics of the turn are concerned, since it produces an increase in the minimum radius of curvature, a diminution in the maximum angular velocity, and in the maximum bank angle.

Translated by R. H. Cramer
Cornell Aeronautical Laboratory, Inc.
Buffalo, New York

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TABLE I

SUMMARY FORMULAE FOR A TURN IN THE CASE OF A PARABOLIC POLAR

Item	Turn				Level flight		
	General formulae	Special cases			With min. amount of thrust	With given amount of thrust	
		φ_{max}	ω_{max}	r_{min}		V_{max}	V_{min}
$x = \frac{C_p}{C_{pe}}$	x	1	$\sqrt{2t - 1}$	$\sqrt{2t^2 - 1}$	1	$t - \sqrt{t^2 - 1}$	$t + \sqrt{t^2 - 1}$
$\frac{V}{V_e}$	$\sqrt{\frac{2t}{1 + x^2}}$	$\sqrt{t}$	1	$\frac{1}{\sqrt{t}}$	1	$\left(t + \sqrt{t^2 - 1}\right)^{\frac{1}{2}}$	$\left(t - \sqrt{t^2 - 1}\right)^{\frac{1}{2}}$
n	$\frac{2tx}{1 + x^2}$	t	$\sqrt{2t - 1}$	$\sqrt{2 - \frac{1}{t^2}}$	1	1	1
$\tan \varphi$	$\sqrt{\left(\frac{2tx}{1 + x^2}\right)^2 - 1}$	$\sqrt{t^2 - 1}$	$\sqrt{2(t - 1)}$	$\sqrt{1 - \frac{1}{t^2}}$	0	0	0
$\frac{\omega V_e}{g}$	$\sqrt{\frac{2tx^2}{1 + x^2} - \frac{1 + x^2}{2t}}$	$\sqrt{t - \frac{1}{t}}$	$\sqrt{2(t - 1)}$	$\sqrt{t - \frac{1}{t}}$	0	0	0
$\frac{rg}{V_e^2}$	$\left[x^2 - \left(\frac{1 + x^2}{2t}\right)^2\right]^{-\frac{1}{2}}$	$\left(1 - \frac{1}{t^2}\right)^{-\frac{1}{2}}$	$\left[2(t - 1)\right]^{-\frac{1}{2}}$	$(t^2 - 1)^{-\frac{1}{2}}$	∞	∞	∞

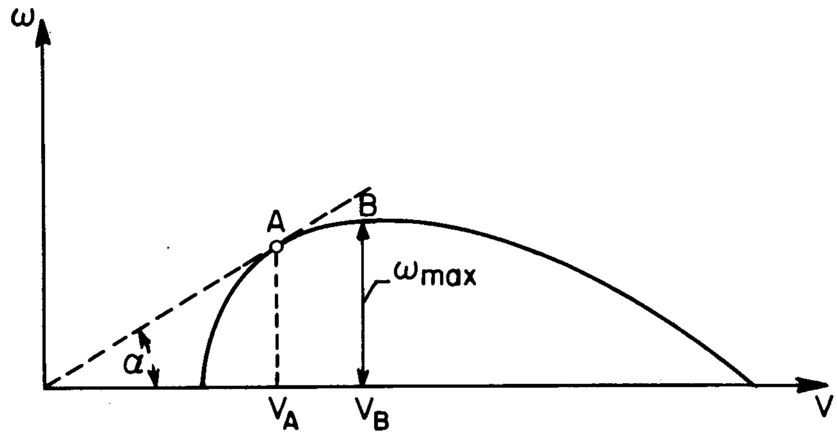

Figure 1.- Graphical determination of the velocity corresponding to the turn made with minimum radius.

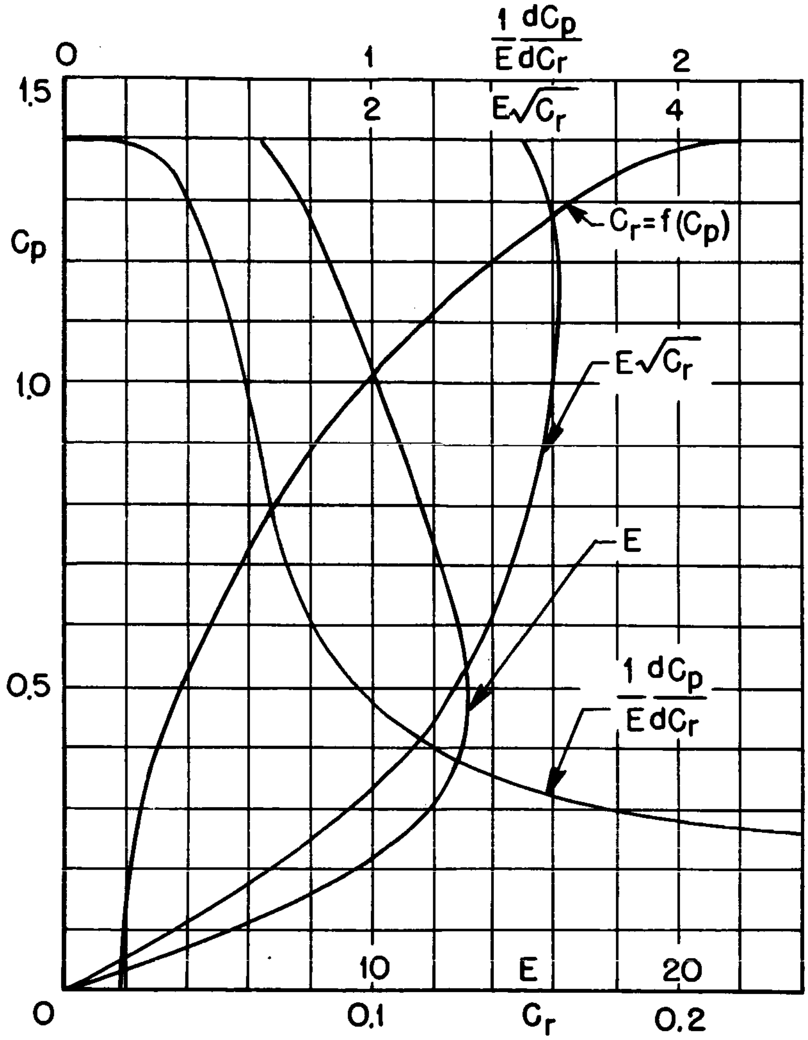

Figure 2.- Polar for the lower velocity range applying to a jet-propelled airplane.

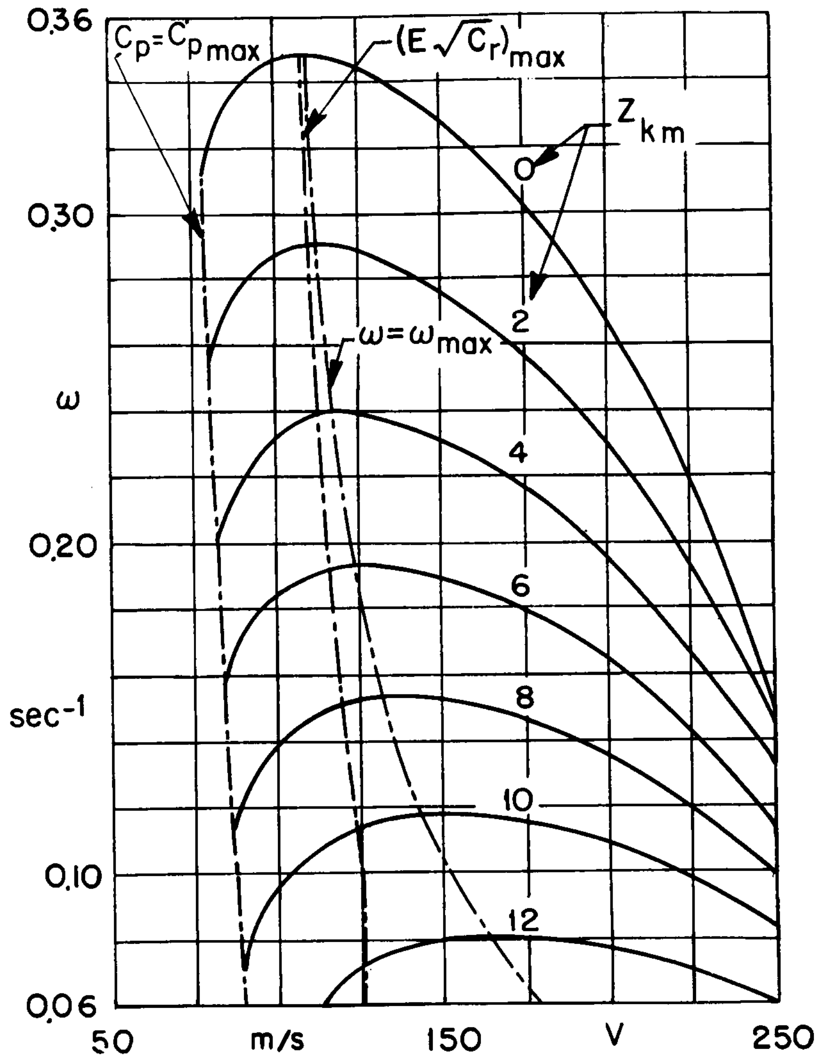

Figure 3.- The angular velocity with which a turn is executed, given as a function of the velocity maintained during the maneuver and of the altitude of flight.

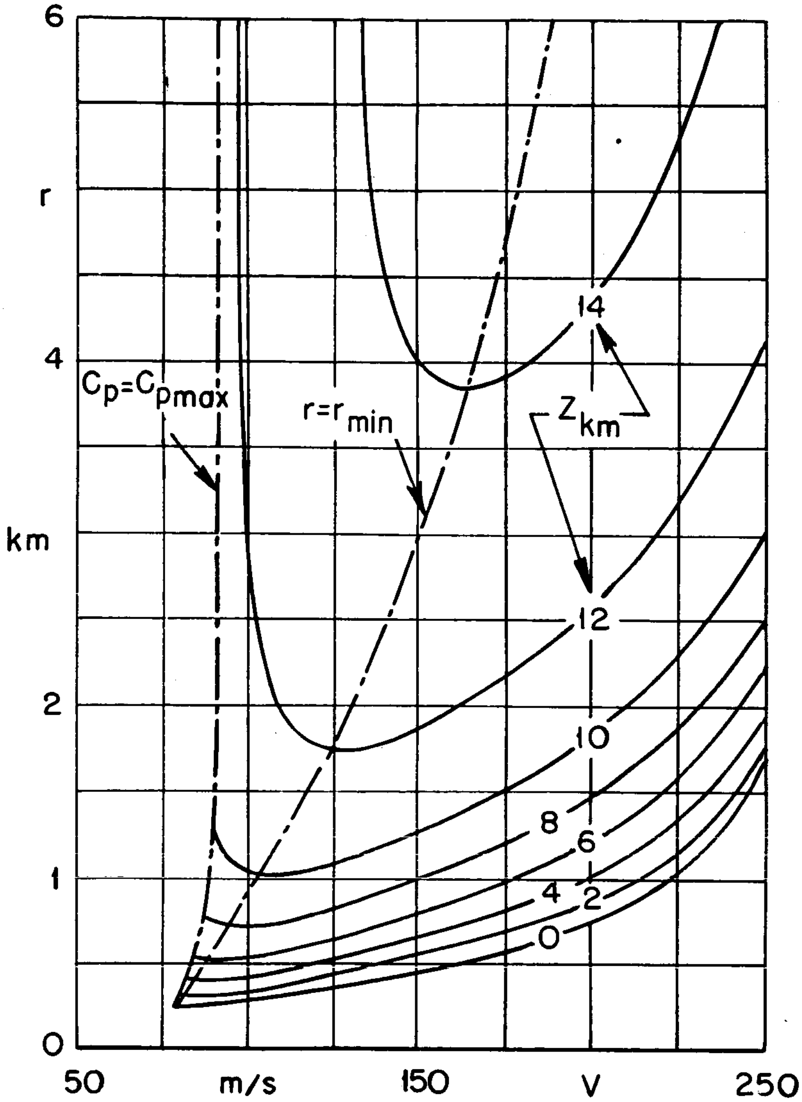

Figure 4.- The radius of turn, given as a function of the velocity maintained during the maneuver and the altitude of flight.

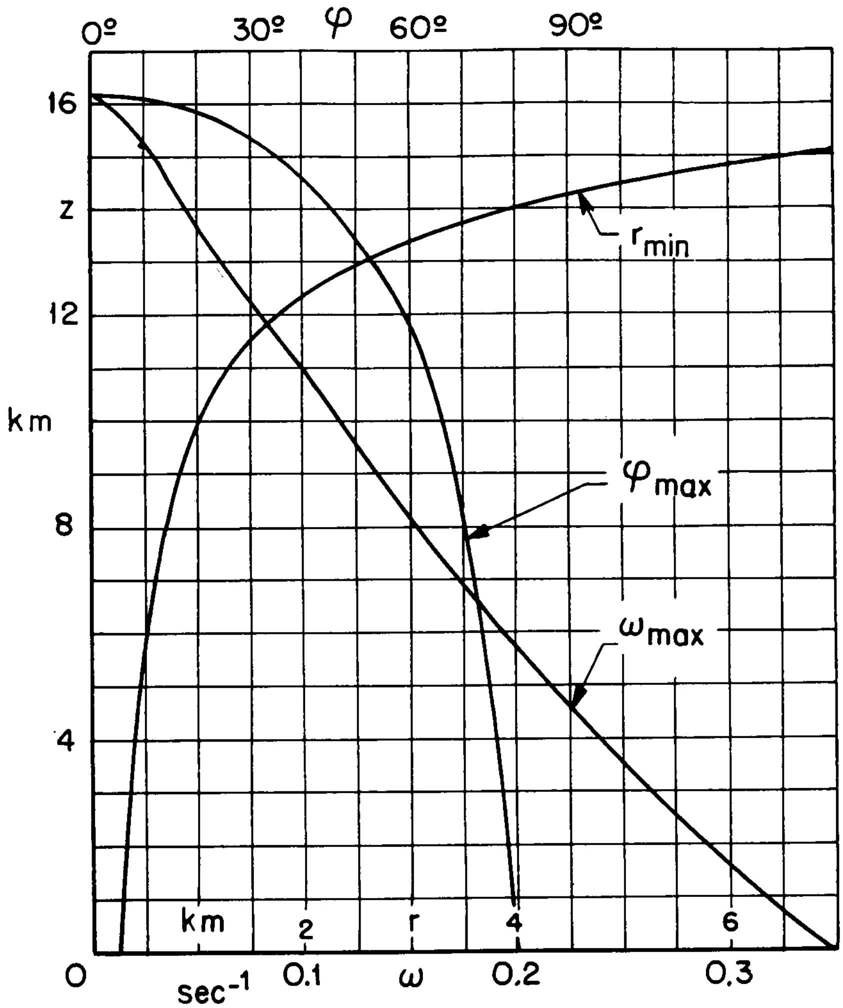

Figure 5.- Typical variation with altitude of the maximum angular velocity, the maximum angle of bank, and the minimum radius of curvature.

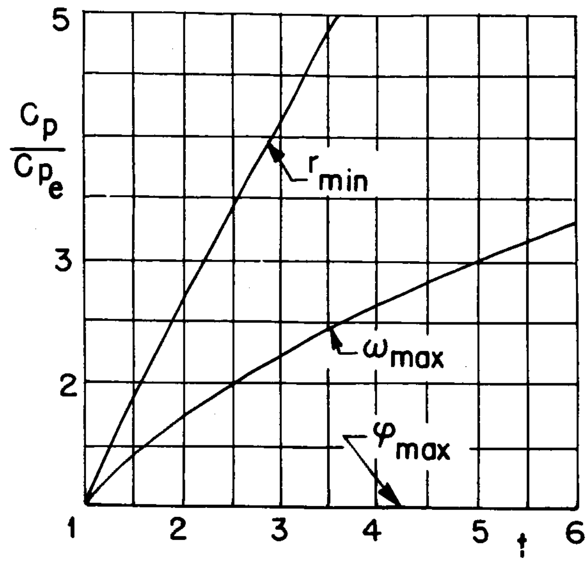

Figure 6.- Lift coefficients pertaining to some special cases of turn.

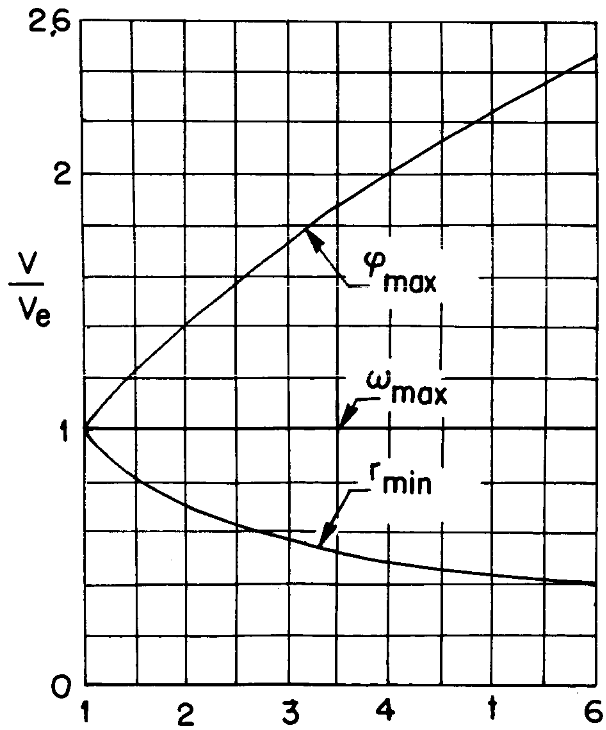

Figure 7.- Velocity maintained during the maneuver for some special cases of turn.

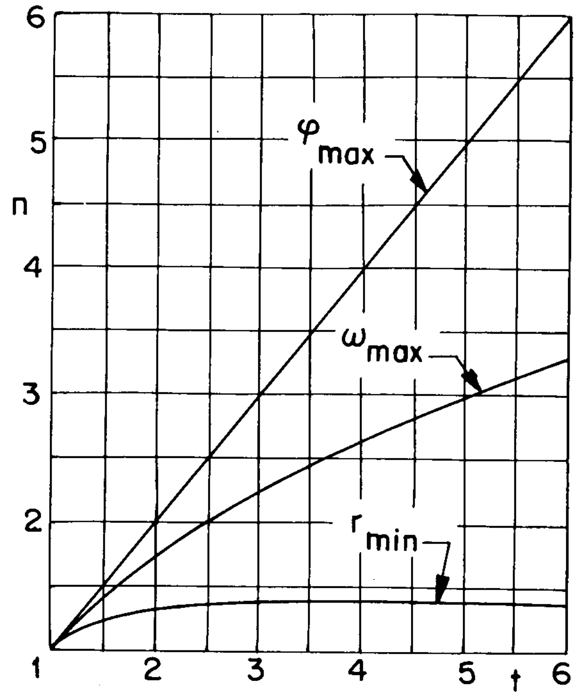

Figure 8.- Load factors in some special cases of turn.

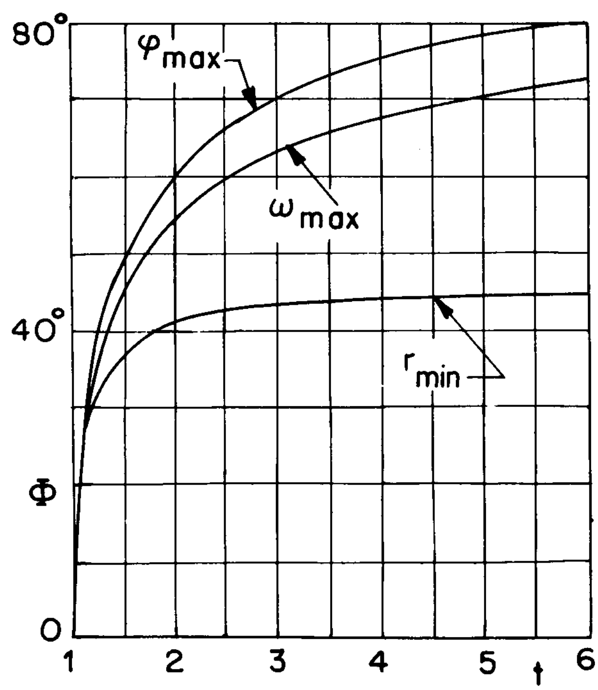

Figure 9.- Angle-of-bank in some special cases of turn.

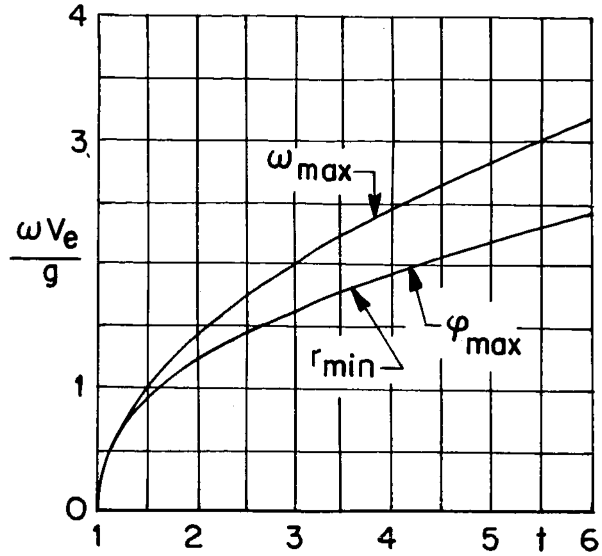

Figure 10.- Angular velocity in some special cases of turn.

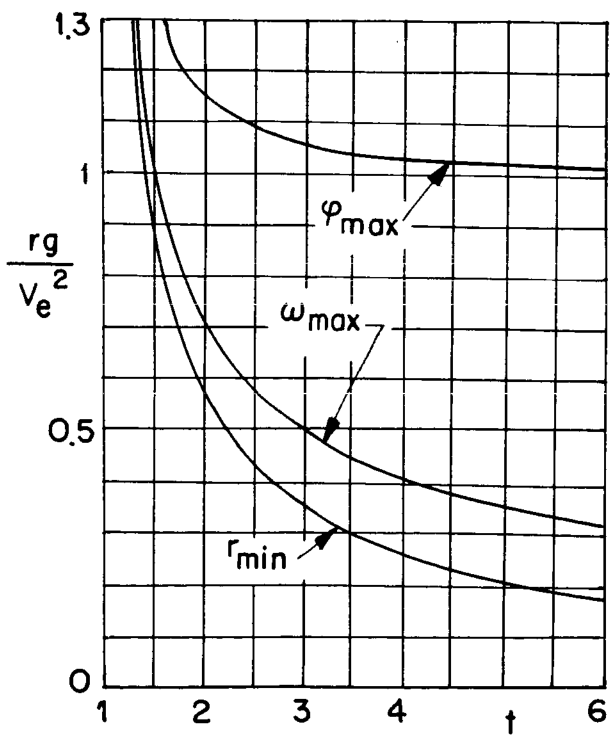

Figure 11.- Radius-of-curvature in some special cases of turn.

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